Math 1510 Week 5

Higher derivatives

For $n \geq 0$, define the n-th derivative of $y = f(x)$ to be

$$\frac{d^n y}{dx^n} = \underbrace{\frac{d}{dx}\left(\cdots \frac{d}{dx}\left(\frac{dy}{dx}\right) \cdots \right)}_{\text{differentiate } n \text{ times}}$$

Other notations:

$$\frac{d^n y}{dx^n} = y^{(n)} = f^{(n)}(x)$$

Second derivative:

$$\frac{d^2 y}{dx^2} = y^{(2)} = y'' = f''(x) = f^{(2)}(x)$$

<u>eg1</u> $y = x^2 + 3x + 7$

$y' = 2x + 3$

$y'' = 2$

$y^{(n)} = 0$ for $n \geq 3$

<u>eg2</u> $f(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} \sin x & \text{if } n = 4m \\ \cos x & \text{if } n = 4m+1 \\ -\sin x & \text{if } n = 4m+2 \\ -\cos x & \text{if } n = 4m+3 \end{cases} \quad \text{for } m \geq 0$$

<u>eg3</u> $f(x) = \frac{1}{x}$ $\qquad f^{(n)}(1) = ?$

<u>Sol</u> $f'(x) = -\frac{1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$ $\quad f'''(x) = -\frac{6}{x^4}$

Similarly $f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$

$$\Rightarrow f^{(n)}(1) = (-1)^n n!$$

Chain Rule

Let f be differentiable at a

g be differentiable at $f(a)$

Then $g \circ f$ is differentiable at a,

$$\boxed{(g\circ f)'(a) = g'(f(a))\, f'(a)}$$

<u>Rmk</u> $(g\circ f)(x) = g(f(x))$

- f is called the inner function
- g is called the outer function

$$\begin{tikzcd} a \arrow[r, mapsto, "f"] \arrow[rr, bend right, "g\circ f"'] & f(a) \arrow[r, mapsto, "g"] & (g\circ f)(a) \end{tikzcd}$$

<u>eg 1</u> $\dfrac{d}{dx}(1+x-x^2)^{10}$

<u>Sol</u> Let $f(x) = 1+x-x^2 \qquad g(u) = u^{10}$

then $f'(x) = 1-2x \qquad g'(u) = 10u^9$

$$\begin{aligned} \therefore \frac{d}{dx}(1+x-x^2)^{10} &= (g\circ f)'(x) \\ &= g'(f(x))\, f'(x) \\ &= \underbrace{10(1+x-x^2)^9}_{\text{derivative of outer function}}\ \underbrace{(1-2x)}_{\text{derivative of inner function}} \end{aligned}$$

Chain Rule in another form (in terms of variables)

Input : x

Intermediate variable : $u = f(x)$

Output : $y = g(u) = (g\circ f)(x)$

Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

eg 2 $\frac{d}{dx}\frac{1}{\sqrt{\log_2 x + \csc x}}$

Sol Let $u = \log_2 x + \csc x$

$$y = \frac{1}{\sqrt{\log_2 x + \csc x}} = \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \left(-\frac{1}{2}u^{-\frac{3}{2}}\right)\left(\frac{1}{x\ln 2} - \csc x \cot x\right)$$

$$= -\frac{1}{2}\underbrace{(\log_2 x + \csc x)}_{\text{derivative of outer function}}{}^{-\frac{3}{2}}\underbrace{\left(\frac{1}{x\ln 2} - \csc x \cot x\right)}_{\text{derivative of inner function}}$$

In practice, we can skip all the steps above.

eg 3 $(5^{\tan x}\sin^4 x)'$

$$= (5^{\tan x})'\sin^4 x + 5^{\tan x}(\sin^4 x)'$$

$$= (\ln 5)\,5^{\tan x}\sec^2 x \sin^4 x$$

$$+ 5^{\tan x}(4\sin^3 x \cos x)$$

Skipped details :

Let $u = \tan x$

$$y = 5^{\tan x} = 5^u$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= (\ln 5)\,5^u \sec^2 x$$

$$= (\ln 5)\,5^{\tan x}\sec^2 x$$

Let $u = \sin x$

$$y = \sin^4 x = u^4$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= 4u^3 \cos x$$

$$= 4\sin^3 x \cos x$$

Repeated Use of Chain Rule

$$(f\circ g\circ h)'(x) = f'(g\circ h(x))\,(g\circ h)'(x)$$
$$= f'(g(h(x)))\,g'(h(x))\,h'(x)$$

eg Consider $\sin\circ\sqrt{}\circ(x^2+\pi x+10)$

$$\frac{d}{dx}\sin\left(\sqrt{x^2+\pi x+10}\right)$$
$$= \underbrace{\cos\sqrt{x^2+\pi x+10}\cdot\frac{1}{2\sqrt{x^2+\pi x+10}}\cdot(2x+\pi)}_{\text{Differentiate layer by layer}}$$

Another form: $v = x^2+\pi x+10$

$$u = \sqrt{x^2+\pi x+10} = \sqrt{v}$$
$$y = \sin\left(\sqrt{x^2+\pi x+10}\right) = \sin u$$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$

Repeated Use of Product Rule

① $(fgh)' = (fg)'h + (fg)h'$
$$= (f'g+fg')h + fgh'$$
$$= f'gh + fg'h + fgh'$$

Ex Find $[x^{10}(\sec x)\ln x]'$

Ans: $x^9\sec x\,(10\ln x + x\tan x\ln x + 1)$

② $(fg)' = f'g + fg'$ 1 1

$(fg)'' = f''g + f'g' + f'g' + fg''$

$\quad = f''g + 2f'g' + fg''$ 1 2 1

$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$ 1 3 3 1

$$\boxed{\text{Leibniz}\quad (fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(k)} g^{(n-k)}}$$

Derivative of Inverses

Let f and f^{-1} be inverses

Then $f(f^{-1}(x)) = x$

$$f'(f^{-1}(x))(f^{-1})'(x) = 1 \text{ by chain rule}$$

$$\Rightarrow \boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

Alternatively, let $y = f^{-1}(x)$, $x = f(y)$

$$x \xmapsto{f^{-1}} y \xmapsto{f} x$$

then $1 = \dfrac{dx}{dx} = \dfrac{dx}{dy} \cdot \dfrac{dy}{dx}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

eg Pf of some basic derivatives:

① $(e^x)' = e^x$

Let $y = e^x$

then $x = \ln y$ $\dfrac{dx}{dy} = \dfrac{1}{y}$ (Proved from definition)

$$\therefore \frac{d}{dx} e^x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{y}} = y = e^x$$

② $(\arctan x)' = \dfrac{1}{1+x^2}$

Let $y = \arctan x$, then $x = \tan y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$$

$$\therefore \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

③ $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

Let $y = \arcsin x$, then $x = \sin y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} \overset{*}{=} \frac{1}{\sqrt{1-\sin^2 y}}$$

$*$ $\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1-\sin^2 y}$

$y = \arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos y \geq 0$

$\therefore \cos y = \sqrt{1-\sin^2 y}$

Ex Prove $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

$(\operatorname{arcsec} x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

Rmk

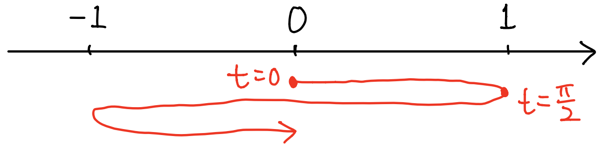
$\operatorname{arcsec}: (-\infty, -1] \cup [1, \infty) \longrightarrow \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Physical Meaning of derivatives

$\dfrac{dy}{dx}$ = Rate of change of y relative to x

eg Let $x(t)$ = displacement at time t

$= \sin t$



Moves between ± 1

$x'(t) = v(t)$ = velocity $= \cos t$

$x''(t) = a(t)$ = acceleration $= -\sin t$

eg

If P = population, $\dfrac{dP}{dt}$ = growth rate

If Q = charge, $\dfrac{dQ}{dt}$ = current

Rate of Change

eg Suppose a spherical snowball melts at a rate of $2\,cm^3/min$. Find the rate of change of its radius and surface area when its radius is $6\,cm$.

Sol Let t be the time

Let the snowball have radius r, surface area S, volume V

Then $\frac{dV}{dt}$ = rate of change of V

$$= -2$$

how V changes relative to r

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$$

$$-2 = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{-1}{2\pi r^2}$$

$$\left.\frac{dr}{dt}\right|_{r=6} = \frac{-1}{2\pi(6)^2} = -\frac{1}{72\pi}\ cm/min$$

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dr}\cdot\frac{dr}{dt}$$

$$= 8\pi r\left(\frac{-1}{2\pi r^2}\right)$$

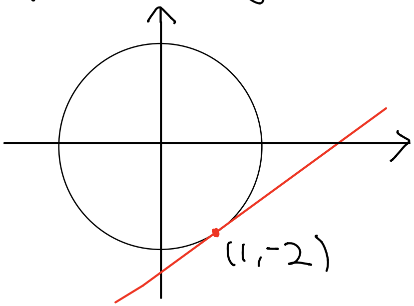
$$= \frac{-4}{r}$$

$$\left.\frac{dS}{dt}\right|_{r=6} = \frac{-4}{6} = -\frac{2}{3}\ cm^2/min$$

Implicit differentiation

eg Consider circle $\mathcal{C}: x^2+y^2=5$

Find equation of tangent at $(1,-2)$



Sol Method I (Express y in terms of x)

$$x^2+y^2=5 \Rightarrow y=\pm\sqrt{5-x^2}$$

Near $(1,-2)$, $y<0$

$$\therefore \quad y=-\sqrt{5-x^2}=-(5-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=-\frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x)$$

$$=x(5-x^2)^{-\frac{1}{2}}$$

$$\left.\frac{dy}{dx}\right|_{(1,-2)}=(1)(5-1^2)^{-\frac{1}{2}}=\frac{1}{2}$$

$\therefore$ Equation of tangent : $y=\frac{1}{2}(x-1)-2$

Sol Method II (Implicit differentiation)

$$x^2+y^2=5$$

Apply $\frac{d}{dx}$ on both sides :

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(5)$$

$$2x+2y\frac{dy}{dx}=0 \qquad \left(\frac{dy^2}{dx}=\frac{dy^2}{dy}\cdot\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(1,-2)}=-\frac{1}{-2}=\frac{1}{2} \Rightarrow \text{Same answer}$$

eg Let $\mathscr{C}: x^3+y^3-9xy=0$

① Show that $(2,4)$ is on $\mathscr{C}$

② Find equation of tangent at $(2,4)$

Sol ① Put $(2,4)$ into $\mathscr{C}$:

$$\text{L.H.S.} = 2^3+4^3-9(2)(4) = 0 = \text{R.H.S.}$$

$\therefore$ $(2,4)$ is on $\mathscr{C}$.

② Apply $\frac{d}{dx}$ to $x^3+y^3-9xy=0$

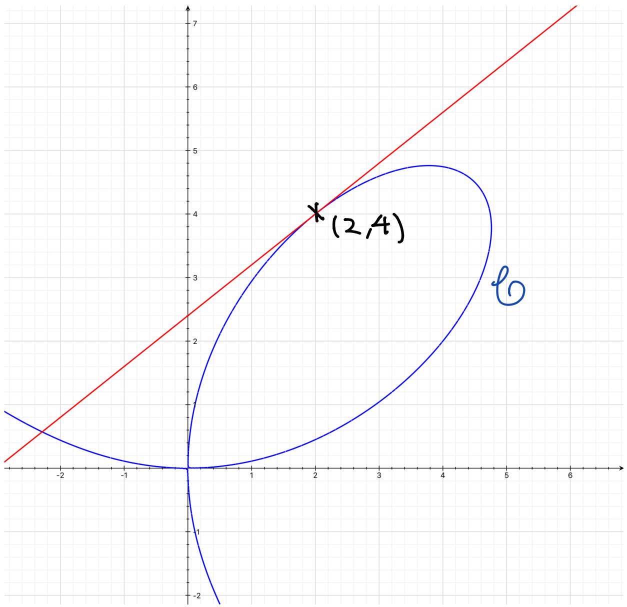
$$3x^2+3y^2\frac{dy}{dx}-9y-9x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{9y-3x^2}{3y^2-9x}=\frac{3y-x^2}{y^2-3x}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(2,4)}=\frac{3(4)-2^2}{4^2-3(2)}=\frac{8}{10}=\frac{4}{5}$$

Tangent: $y=\frac{4}{5}(x-2)+4$

$$4x-5y+12=0$$

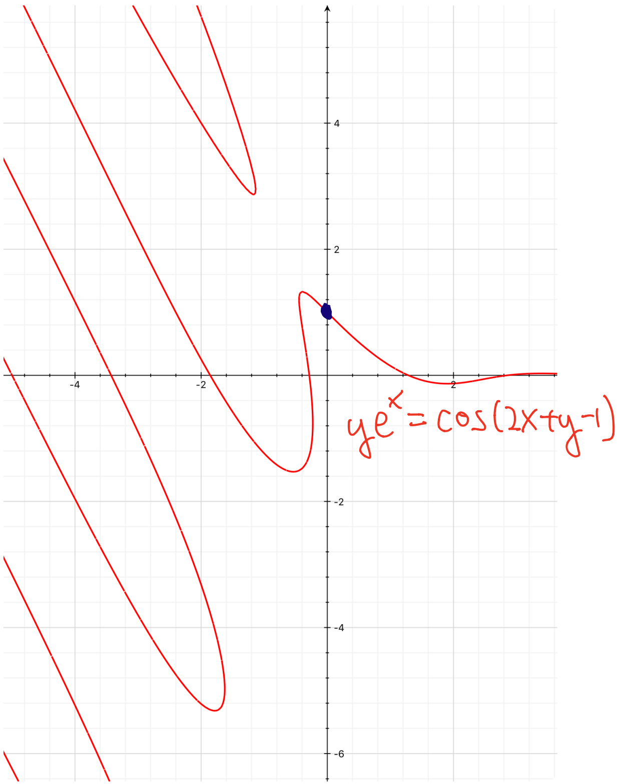


eg Suppose $ye^x = \cos(2x+y-1)$. Find y' and y'' at $(x,y)=(0,1)$

Sol $\frac{d}{dx}$ the given equation twice

$$y'e^x + ye^x = -\sin(2x+y-1)\,(2+y') \quad \cdots\cdots \; (1)$$

$$y''e^x + y'e^x + y'e^x + ye^x = -\cos(2x+y-1)\,(2+y')^2 - \sin(2x+y-1)\,y'' \quad \cdots \; (2)$$



Put $(x,y)=(0,1)$ into (1)

$$y'(1) + 1 = 0$$

$$y'\big|_{(0,1)} = -1 \quad \cdots\cdots \; (3)$$

Put $(x,y)=(0,1)$, (3) into (2)

$$y'' - 1 - 1 + 1 = -(1)(2-1)^2 - 0$$

$$y''\big|_{(0,1)} = 0$$

Rmk Hard to express y as a function of x

Logarithmic Differentiation

eg Find y', where $y=(2+\sin x)^{\tan x}$

Recall:

$$\frac{d}{dx}x^a = a x^{a-1}$$

$$\frac{d}{dx}a^x = (\ln a)a^x$$

either base or exponent is a constant

$\therefore$ Cannot apply directly

Sol Method 1 (Trick: $y = e^{\ln y}$)

$$y = e^{\ln y} = e^{\tan x \ln(2+\sin x)}$$

base is constant

$$y' = e^{\tan x \ln(2+\sin x)}\left(\sec^2 x \ln(2+\sin x) + \frac{\tan x \cos x}{2+\sin x}\right)$$

$$= (2+\sin x)^{\tan x}\left(\sec^2 x \ln(2+\sin x) + \frac{\sin x}{2+\sin x}\right)$$

Method 2 (Logarithmic Differentiation)

$$y = (2+\sin x)^{\tan x}$$

$$\ln y = \tan x \ln(2+\sin x)$$

Apply $\frac{d}{dx}$

$$\frac{1}{y}y' = \sec^2 x \ln(2+\sin x) + \frac{\tan x \cos x}{2+\sin x}$$

$$y' = y\left(\sec^2 x \ln(2+\sin x) + \frac{\sin x}{2+\sin x}\right)$$

$$= (2+\sin x)^{\tan x}\left(\sec^2 x \ln(2+\sin x) + \frac{\sin x}{2+\sin x}\right)$$

Rmk Method 1 and 2 use same idea, just different presentations eg Let $y=\sqrt{\dfrac{(x-1)^3(x^2+6)}{(2x+7)^5}}$, $x>1$

Find y' by logarithmic differentiation

Sol

$$\ln y=\frac{1}{2}\left[3\ln(x-1)+\ln(x^2+6)-5\ln(2x+7)\right] \quad (*)$$

$\frac{d}{dx}$:

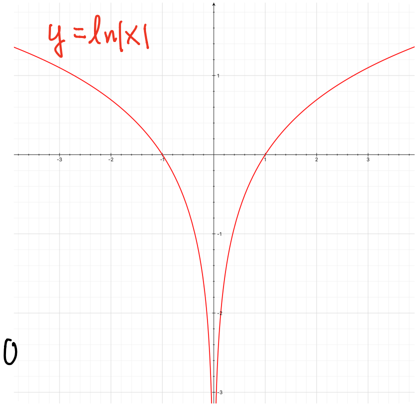
$$\frac{1}{y}y'=\frac{1}{2}\left(\frac{3}{x-1}+\frac{2x}{x^2+6}-\frac{10}{2x+7}\right)$$

$$y'=\frac{y}{2}\left(\frac{3}{x-1}+\frac{2x}{x^2+6}-\frac{10}{2x+7}\right)$$

Rmk

$x>1$ ensures that

$(*)$ is defined



More generally,

$$\frac{d}{dx}\ln|x|=\frac{1}{x} \quad \text{for } x\neq 0$$

Pf of Power rule by log differentiation

Let $y=x^a$, $x>0$

then $\ln y=\ln x^a=a\ln x$

$\frac{d}{dx}$:

$$\frac{1}{y}y'=a\cdot\frac{1}{x}$$

$$y'=\frac{ay}{x}$$

$$=\frac{ax^a}{x}$$

$$\therefore \quad \frac{d}{dx}x^a=ax^{a-1}$$

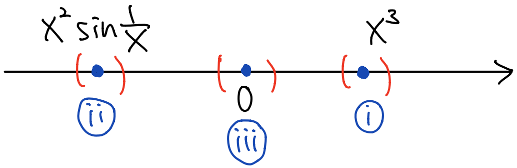
Derivatives of Piecewise functions

eg

$$\text{Let } f(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ x^2 \sin\frac{1}{x} & \text{if } x < 0 \end{cases}$$

Find $f'(x)$ and $f''(x)$

Sol



(i) For $x > 0$,

$f(x) = x^3$ (and same formula nearby)

$f'(x) = 3x^2$

(ii) Similarly, for $x < 0$

$f(x) = x^2 \sin\frac{1}{x}$ (and nearby)

$\Rightarrow f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

(iii) Near 0, $f(x)$ is not defined by a single formula.

$\therefore$ Need to find $f'(0)$ from definition

$Lf'(0)$

$$= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{h^2\sin\frac{1}{h} - 0^3}{h}$$

$$= \lim_{h\to 0^-} h\sin\frac{1}{h}$$

$= 0$ (Sandwich thm)

$Rf'(0)$

$$= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{h^3 - 0^3}{h}$$

$$= \lim_{h\to 0^+} h^2$$

$= 0$

$Lf'(0) = Rf'(0) = 0 \Rightarrow f'(0)$ exists and $f'(0) = 0$

Warning Wrong way to compute $f'(0)$:

$f(x) = x^3$ for $x \geq 0 \;\cancel{\Longrightarrow}\; f'(x) = 3x^2$ for $x \geq 0$

$\Longrightarrow f'(0) = 3(0)^2 = 0$

$$\therefore f'(x) = \begin{cases} 3x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x < 0 \end{cases}$$

By similar argument (Exercise)

$$f''(x) = \begin{cases} 6x & \text{if } x > 0 \\ \text{DNE} & \text{if } x = 0 \\ \left(2 - \frac{1}{x^2}\right)\sin\frac{1}{x} - \frac{2}{x}\cos\frac{1}{x} & \text{if } x < 0 \end{cases}$$

<u>Rmk</u> Another way to show $f''(0)$ DNE:

$f'(x)$ is not continuous at 0

$\Rightarrow f'(x)$ is not differentiable at 0

$\Rightarrow f''(0)$ DNE